

MOTIVIC SERRE INVARIANTS MODULO THE SQUARE OF $\mathbb{L} - 1$

TAKEHIKO YASUDA

ABSTRACT. Motivic Serre invariants defined by Loeser and Sebag are elements of the Grothendieck ring of varieties modulo $\mathbb{L} - 1$. In this paper, we show that we can lift these invariants to modulo the square of $\mathbb{L} - 1$ after tensoring the Grothendieck ring with \mathbb{Q} , under certain assumptions.

1. INTRODUCTION

Let K be a complete discrete valuation field with a perfect residue field k . For a smooth projective irreducible K -variety X , Loeser and Sebag [9] defined the motivic Serre invariant $S(X)$. This invariant belongs to the ring $K_0(\text{Var}_k)/(\mathbb{L} - 1)$, where $K_0(\text{Var}_k)$ is the Grothendieck ring of k -varieties and $\mathbb{L} := [\mathbb{A}_k^1]$, the class of an affine line in this ring. Let $K_0(\text{Var}_k)_{\mathbb{Q}} := K_0(\text{Var}_k) \otimes_{\mathbb{Z}} \mathbb{Q}$. In this paper, we construct, under a certain assumption, an invariant

$$\tilde{S}(X) \in K_0(\text{Var}_k)_{\mathbb{Q}}/(\mathbb{L} - 1)^2$$

which coincides with $S(X)$ in $K_0(\text{Var}_k)_{\mathbb{Q}}/(\mathbb{L} - 1)$.

Remark 1.1. Loeser and Sebag defined the motivic Serre invariant more generally for smooth quasi-compact separated rigid K -spaces. For the sake of simplicity, we consider only the case where X is a projective variety.

Let \mathcal{O} be the valuation ring of K . The assumption we will make is that the desingularization theorem and the weak factorization theorem hold, their precise statements are as follows:

- Assumption 1.2.** (1) (*Desingularization*) *There exists a regular projective flat \mathcal{O} -scheme \mathcal{X} with the generic fiber $\mathcal{X}_K := \mathcal{X} \otimes_{\mathcal{O}} K = X$ such that the special fiber $\mathcal{X}_k := \mathcal{X} \otimes_{\mathcal{O}} k$ is a simple normal crossing divisor in \mathcal{X} . (We call such an \mathcal{X} a regular snc model of X .)*
- (2) (*Weak factorization*) *Let \mathcal{X} and \mathcal{X}' be regular snc models of X . Then there exist finitely many regular snc models of X ,*

$$\mathcal{X}_0 = \mathcal{X}, \mathcal{X}_1, \dots, \mathcal{X}_n = \mathcal{X}',$$

2010 *Mathematics Subject Classification.* 14D06, 14E05.

The most part of this work was done during the author's stay at Institut des Hautes Études Scientifiques. He is grateful for its hospitality and great environment. He also wishes to thank François Loeser for inspiring discussion and helpful comments. This work was partly supported by JSPS KAKENHI Grant Number JP15K17510 and JP16H06337.

such that for every i , either the birational map $\mathcal{X}_i \dashrightarrow \mathcal{X}_{i+1}$ is the blowup along a regular center $Z \subset \mathcal{X}_{i+1,k}$ which has normal crossings¹ with \mathcal{X}_k or its inverse $\mathcal{X}_{i+1} \dashrightarrow \mathcal{X}_i$ has the same description with $\mathcal{X}_{i+1,k}$ replaced with $\mathcal{X}_{i,k}$.

When X has dimension one, this assumption holds as is well-known. Indeed the above desingularization theorem in this case follows from the desingularization theorem for excellent surfaces by Abhyankar, Hironaka and Lipman (see [8]), while the weak factorization follows from the fact that every proper birational morphism of regular integral noetherian schemes of dimension two factors into a sequence of finitely many blowups at closed points. The last fact is well-known in the case of varieties over an algebraically closed field (for instance, [5, V, Cor. 5.4]) and is valid even in our situation as proved in [7, Th. 4.1] in a more general context. Assumption 1.2 holds also when k has characteristic zero. This follows from the recent generalizations to excellent schemes respectively by Temkin [12, 13] and by Abramovich and Temkin [2] of the Hironaka desingularization theorem and the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1].

Let \mathcal{X} be a regular snc model of X , let \mathcal{X}_{sm} be its \mathcal{O} -smooth locus and let $\mathcal{X}_{\text{sm},k} := \mathcal{X}_{\text{sm}} \otimes_{\mathcal{O}} k$. Then \mathcal{X}_{sm} is a weak Neron model of X in the sense of [3] and by definition,

$$S(X) = [\mathcal{X}_{\text{sm},k}] \in K_0(\text{Var}_k)/(\mathbb{L} - 1).$$

To define our invariant $\tilde{S}(X)$, we also need information on the non-smooth locus of \mathcal{X} . Regard \mathcal{X}_k as a divisor and write it as $\mathcal{X}_k = \sum_{i \in I} a_i D_i$, where D_i are the irreducible components of \mathcal{X}_k and a_i are the multiplicities of D_i in \mathcal{X} respectively. For a subset $H \subset I$, we define

$$D_H^\circ := \bigcap_{h \in H} D_h \setminus \bigcup_{i \in I \setminus H} D_i.$$

When $H = \{i\}$, we abbreviate it to D_i° , and when $H = \{i, j\}$, to D_{ij}° . These locally closed subsets give the stratification

$$\mathcal{X}_k = \bigsqcup_{\emptyset \neq H \subset I} D_H^\circ$$

and the stratification

$$\mathcal{X}_{\text{sm},k} = \bigcup_{i \in I: a_i=1} D_i^\circ.$$

From the second stratification, we see

$$S(X) = \sum_{i \in I: a_i=1} [D_i^\circ] \in K_0(\text{Var}_k)/(\mathbb{L} - 1).$$

Loeser and Sebag proved in the paper cited above that this is independent of the model \mathcal{X} and depends only on X .

¹That Z has normal crossings with $\mathcal{X}_{i+1,k}$ means that for every closed point $x \in \mathcal{X}_{i+1,k}$, there exist a regular system of parameters $x_1, \dots, x_d \in \mathcal{O}_{\mathcal{X}_{i+1},x}$ such that in an open neighborhood of x , the support of the special fiber $\mathcal{X}_{i+1,k}$ is the zero locus of $\prod_{v \in V} x_v$ for some subset $V \subset \{1, \dots, d\}$ and Z is the common zero locus of x_w , $w \in W$ for some $W \subset \{1, \dots, d\}$.

Definition 1.3. For a regular snc model \mathcal{X} of X , we define

$$\tilde{S}(\mathcal{X}) := \sum_{i \in I: a_i=1} [D_i^\circ] + \sum_{\substack{\{i,j\} \subset I: \\ (a_i, a_j)=1}} \frac{1}{a_i a_j} [D_{ij}^\circ] (1 - \mathbb{L})$$

as an element of $K_0(\text{Var}_k)_{\mathbb{Q}}/(\mathbb{L} - 1)^2$. Here (a, b) denotes the greatest common divisor of a and b .

Obviously, the two invariants $S(X)$ and $\tilde{S}(\mathcal{X})$ coincide when they are sent to $K_0(\text{Var}_k)_{\mathbb{Q}}/(\mathbb{L} - 1)$ by the natural maps.

The following is our main theorem:

Theorem 1.4. *Let X be a smooth projective K -variety. Under Assumption 1.2, the invariant $\tilde{S}(\mathcal{X})$ is independent of the chosen regular snc model \mathcal{X} and depends only on X .*

The theorem allows us to think of $\tilde{S}(\mathcal{X})$ as an invariant of X and denote it by $\tilde{S}(X)$, which is what was mentioned at the beginning of this Introduction.

2. PREPARATORY REDUCTIONS

We generalize the invariant $\tilde{S}(\mathcal{X})$ as follows. Let \mathcal{X} be a regular flat \mathcal{O} -scheme of finite type such that \mathcal{X}_K is smooth and $\mathcal{X}_k = \bigcup_{i \in I} D_i$ is a simple normal crossing divisor in \mathcal{X} . (We no longer suppose that \mathcal{X} or \mathcal{X}_K is projective.) For a constructible subset $C \subset \mathcal{X}_k$, we define

$$\tilde{S}(\mathcal{X}, C) := \sum_{\substack{i \in I: \\ a_i=1}} [D_i^\circ \cap C] + \sum_{\substack{\{i,j\} \subset I: \\ (a_i, a_j)=1}} \frac{1}{a_i a_j} [D_{ij}^\circ \cap C] (1 - \mathbb{L})$$

as an element of $K_0(\text{Var}_k)_{\mathbb{Q}}/(\mathbb{L} - 1)^2$.

Let $f: \mathcal{Y} \rightarrow \mathcal{X}$ the blowup along a smooth irreducible center $Z \subset \mathcal{X}_k$ which has normal crossings with \mathcal{X}_k . Then, \mathcal{Y} is an \mathcal{O} -scheme satisfying the same conditions as \mathcal{X} does and we can similarly define $\tilde{S}(\mathcal{Y}, C')$ for a constructible subset $C' \subset \mathcal{Y}_k$.

Theorem 1.4 follows from:

Proposition 2.1. *Let \mathcal{X} be as above and for any constructible subset $C \subset \mathcal{X}_k$,*

$$\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, f^{-1}(C)).$$

Indeed, Theorem 1.4 is a direct consequence of this proposition with $C = \mathcal{X}_k$ and Assumption 1.2.

In what follows, we will prove this proposition. First we will reduce it to the local situation by using:

Lemma 2.2. (1) *If C is the disjoint union $\bigsqcup_{s=1}^l C_s$ of constructible subsets C_s , then*

$$\tilde{S}(\mathcal{X}, C) = \sum_{s=1}^l \tilde{S}(\mathcal{X}, C_s).$$

- (2) Let $\mathcal{X} = \bigcup_{\lambda \in \Lambda} U_\lambda$ be an open covering. Suppose that for every constructible subset $C \subset \mathcal{X}_k$ and for every $\lambda \in \Lambda$,

$$\tilde{S}(\mathcal{X}, C \cap U_\lambda) = \tilde{S}(\mathcal{Y}, f^{-1}(C \cap U_\lambda)).$$

Then, for every constructible subset $C \subset \mathcal{X}_k$, we have

$$\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, f^{-1}(C)).$$

Proof. The first assertion is obvious. To show the second one, we first claim that there exists a stratification $C = \bigsqcup_{s=0}^n C_s$ with C_s constructible such that each C_s is contained in some U_λ . Indeed we can take C_0 as $C \cap U_\lambda$ such that C and C_0 have equal dimension, then construct C_1 applying the same procedure to $C \setminus U_\lambda$ and so on.

By the assumption, for every s , $\tilde{S}(\mathcal{X}, C_s) = \tilde{S}(\mathcal{Y}, f^{-1}(C_s))$. Now, from the first assertion, we get

$$\tilde{S}(\mathcal{X}, C) = \sum_s \tilde{S}(\mathcal{X}, C_s) = \sum_s \tilde{S}(\mathcal{Y}, f^{-1}(C_s)) = \tilde{S}(\mathcal{Y}, f^{-1}(C)).$$

□

Let $x \in \mathcal{X}_k$ be a closed point and take a local coordinate system $x_1, \dots, x_d \in \mathcal{O}_{\mathcal{X}, x}$. By shrinking \mathcal{X} if necessary, we may suppose that x_1, \dots, x_d are global sections of $\mathcal{O}_{\mathcal{X}}$ and that the special fiber \mathcal{X}_k is the zero locus of $\prod_{i=1}^{d'} x_i$, $d' \leq d$ (thus we identify I with $\{1, \dots, d'\}$) and Z is the common zero locus of x_j , $j \in J$ for some subset $J \subset \{1, \dots, d\}$. From the first assertion of the above lemma, since we obviously have

$$\tilde{S}(\mathcal{X}, C \setminus Z) = \tilde{S}(\mathcal{Y}, f^{-1}(C \setminus Z)),$$

we may also assume that $C \subset Z$. In a few following sections, we will prove Proposition 2.1 in this situation, discussing separately in the cases ($\#I =$) $d' = 1$, $d' = 2$ and $d' \geq 3$. Before that, we prepare some notation and a lemma.

Notation 2.3. For $i \in I$, let D_i be the prime divisor of \mathcal{X} given by $x_i = 0$ and let $E_i \subset \mathcal{Y}_k$ be its strict transform. Let $E_0 \subset \mathcal{Y}_k$ be the exceptional divisor of the blowup $f: \mathcal{Y} \rightarrow \mathcal{X}$. We denote $f^{-1}(C)$ by \tilde{C} .

The multiplicity of E_i in \mathcal{Y}_k is a_i for $i \in I$ and

$$(2.1) \quad a_0 := \sum_{Z \subset D_i} a_i$$

for $i = 0$. We will use the following lemma several times.

Lemma 2.4. *Suppose that $C \subset Z$. Then, for $i \in I \setminus J$, we have $\tilde{C} \subset E_i$.*

Proof. The morphism $\tilde{C} \rightarrow C$ is a $\mathbb{P}^{\#J-1}$ -bundle. The divisor E_i is the blowup of D_i along $Z \cap D_i$, which has codimension $\#J$ in D_i . It follows that $E_i \cap \tilde{C} \rightarrow C$ is also a $\mathbb{P}^{\#J-1}$ -bundle. Hence \tilde{C} and $E_i \cap \tilde{C}$ coincide and the lemma follows. □

3. THE CASE $d' = 1$.

We now begin the proof of Proposition 2.1 in the situation described just before Notation 2.3. In this section, we consider the case $d' = 1$.

Since $Z \subset \mathcal{X}_k$, recalling $I = \{1, \dots, d'\}$, we see that $1 \in J$. Then

$$\tilde{S}(\mathcal{X}, C) = \begin{cases} [C] & (a_1 = 1) \\ 0 & (\text{otherwise}) \end{cases}.$$

From (2.1), $a_0 = a_1$, and $(a_0, a_1) = a_1$. Hence, if $a_1 \neq 1$, then

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = 0 = \tilde{S}(\mathcal{X}, C).$$

If $a_1 = 1$, then recalling that $C \subset Z$, we see that $\tilde{C} \subset E_0 = f^{-1}(Z)$ and that

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = [\tilde{C} \setminus E_1] + [E_1 \cap \tilde{C}](1 - \mathbb{L}).$$

To compute the right hand side of this equality, we first observe that \tilde{C} is a $\mathbb{P}^{\sharp J-1}$ -bundle over C . The divisor E_1 is the blowup of D_1 along Z . Therefore $E_1 \cap \tilde{C}$ is a $\mathbb{P}^{\sharp J-2}$ -bundle over C . Hence

$$\begin{aligned} \tilde{S}(\mathcal{Y}, \tilde{C}) &= [C](\mathbb{P}^{\sharp J-1}) - [C](\mathbb{P}^{\sharp J-2}) + [C](\mathbb{P}^{\sharp J-2})(1 - \mathbb{L}) \\ &= [C](\mathbb{L}^{\sharp J-1} + (1 + \mathbb{L} + \dots + \mathbb{L}^{\sharp J-2})(1 - \mathbb{L})) \\ &= [C](\mathbb{L}^{\sharp J-1} + 1 - \mathbb{L}^{\sharp J-1}) \\ &= [C] \\ &= \tilde{S}(\mathcal{X}, C). \end{aligned}$$

We conclude that if $d' = 1$, then $\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, \tilde{C})$.

4. THE CASE $d' = 2$.

Next we consider the case $d' = 2$. We have

$$C = (C \cap D_1^\circ) \sqcup (C \cap D_2^\circ) \sqcup (C \cap D_{12}^\circ).$$

From the case $\sharp I = 1$ treated in the last section, we have

$$\tilde{S}(\mathcal{X}, C \cap D_i^\circ) = \tilde{S}(\mathcal{Y}, f^{-1}(C \cap D_i^\circ)) \quad (i = 1, 2).$$

Therefore, from Lemma 2.2, replacing C with $C \cap D_{12}^\circ$, we may suppose that $C \subset D_{12}^\circ = D_1 \cap D_2$. Then we have

$$\tilde{S}(\mathcal{X}, C) = \begin{cases} \frac{1}{a_1 a_2} [C](1 - \mathbb{L}) & ((a_1, a_2) = 1) \\ 0 & (\text{otherwise}) \end{cases}.$$

We next compute $\tilde{S}(\mathcal{Y}, \tilde{C})$ separately in the case $Z \subset D_1 \cap D_2$ and in the case $Z \not\subset D_1 \cap D_2$.

In the former case, we have $a_0 = a_1 + a_2 \neq 1$ and

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = \sum_{\substack{i \in \{1, 2\}: \\ (a_0, a_i) = 1}} \frac{1}{a_0 a_i} [\tilde{C} \cap E_{0i}^\circ](1 - \mathbb{L}).$$

If $(a_1, a_2) \neq 1$, then $(a_0, a_1) \neq 1$ and $(a_0, a_2) \neq 1$, which show $\tilde{S}(\mathcal{Y}, \tilde{C}) = 0 = \tilde{S}(\mathcal{X}, C)$.
 If $(a_1, a_2) = 1$, then we have $(a_0, a_1) = (a_0, a_2) = 1$, and

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = \sum_{i=1}^2 \frac{1}{a_0 a_i} [\tilde{C} \cap E_{0i}^\circ] (1 - \mathbb{L}).$$

Since $E_1 \cap \tilde{C} = E_0 \cap E_1 \cap \tilde{C} \rightarrow C$ is a trivial $\mathbb{P}^{\sharp J-2}$ -bundle and $E_1 \cap E_2 \cap \tilde{C} \rightarrow C$ is a hyperplane in it, $E_{01}^\circ \cap \tilde{C} \rightarrow \tilde{C}$ is a trivial $\mathbb{A}^{\sharp J-2}$ -bundle. Similarly for $E_{02}^\circ \cap \tilde{C} \rightarrow \tilde{C}$. Hence

$$\begin{aligned} \tilde{S}(\mathcal{Y}, \tilde{C}) &= \left(\frac{1}{(a_1 + a_2)a_1} + \frac{1}{(a_1 + a_2)a_2} \right) [C] \mathbb{L}^{\sharp J-2} (1 - \mathbb{L}) \\ &= \frac{1}{a_1 a_2} [C] \mathbb{L}^{\sharp J-2} (1 - \mathbb{L}) \\ &\stackrel{\star}{=} \frac{1}{a_1 a_2} [C] (1 - \mathbb{L}) \\ &= \tilde{S}(\mathcal{X}, C). \end{aligned}$$

Here the equality marked with \star follows from

$$\mathbb{L}(1 - \mathbb{L}) = (\mathbb{L} - 1)(1 - \mathbb{L}) + 1 - \mathbb{L} = 1 - \mathbb{L} \pmod{(\mathbb{L} - 1)^2}.$$

In the case $Z \not\subset D_1 \cap D_2$, we have either $Z \subset D_1$ or $Z \subset D_2$. Since the two cases are similar, we only discuss the former case. Since $2 \in I \setminus J$, from Lemma 2.4, we have $\tilde{C} \subset E_0 \cap E_2$. Since $a_0 = a_1$, $\tilde{C} \rightarrow C$ is a $\mathbb{P}^{\sharp J-1}$ -bundle and $\tilde{C} \cap E_1 \rightarrow C$ is a $\mathbb{P}^{\sharp J-2}$ -bundle, we have

$$\begin{aligned} \tilde{S}(\mathcal{Y}, \tilde{C}) &= \frac{1}{a_0 a_2} [\tilde{C} \cap E_{0,2}^\circ] (1 - \mathbb{L}) \\ &= \frac{1}{a_1 a_2} [\tilde{C} \setminus E_1] (1 - \mathbb{L}) \\ &= \frac{1}{a_1 a_2} [C] [\mathbb{P}^{\sharp J-1} \setminus \mathbb{P}^{\sharp J-2}] (1 - \mathbb{L}) \\ &= \frac{1}{a_1 a_2} [C] \mathbb{L}^{\sharp J-1} (1 - \mathbb{L}) \\ &= \frac{1}{a_1 a_2} [C] (1 - \mathbb{L}) \\ &= \tilde{S}(\mathcal{X}, C). \end{aligned}$$

We have completed the proof that $\tilde{S}(\mathcal{Y}, \tilde{C}) = \tilde{S}(\mathcal{X}, C)$, when $d' = 2$.

5. THE CASE $\sharp I \geq 3$.

As in the last section, by induction on $\sharp I$, we may suppose that $C \subset \bigcap_{i \in I} D_i$. Then $\tilde{S}(\mathcal{X}, C) = 0$. On the other hand, $\tilde{S}(\mathcal{Y}, \tilde{C})$ is a \mathbb{Q} -linear combination of

$$A_i := [\tilde{C} \cap E_{0i}^\circ] (1 - \mathbb{L}), \quad i \in I,$$

and

$$B := \delta_{1,a_0} \left[\tilde{C} \cap E_0^\circ \right],$$

with δ_{1,a_0} being the Kronecker delta. Thus it suffices to show that $A_i = 0$, $i \in I$ and that $B = 0$.

We first show that $B = 0$. If $\sharp I \cap J \geq 2$, then

$$a_0 = \sum_{i \in I \cap J} a_i > 1.$$

Hence $B = 0$. If $\sharp(I \cap J) < 2$, then $I \setminus J$ is non-empty and Lemma 2.4 shows that $\tilde{C} \cap E_0^\circ$ is empty, hence $B = 0$.

Next we show that $A_i = 0$. If $\sharp(I \setminus J) \geq 2$, then from Lemma 2.4, for every $i \in I$, there exists $i' \in I \setminus \{i\}$ such that $\tilde{C} \subset E_{i'}$. Hence $\tilde{C} \cap E_{0i}^\circ = \emptyset$ and $A_i = 0$.

If $\sharp(I \setminus J) = 1$, then by the same reasoning as above, $A_i = 0$ for $i \in I \cap J$. For $i \in I \setminus J$,

$$\tilde{C} \cap E_{0i}^\circ = \mathbb{P}_C^{\sharp J - 1} \setminus \bigcup_{j \in I \cap J} H_j,$$

where $\mathbb{P}_C^{\sharp J - 1}$ denotes the trivial $\mathbb{P}^{\sharp J - 1}$ -bundle $\mathbb{P}^{\sharp J - 1} \times C$ over C and H_j are coordinate hyperplanes of $\mathbb{P}_C^{\sharp J - 1}$. Since $\sharp(I \cap J) \geq 2$,

$$A_i = [C][\mathbb{G}_m^{\sharp(I \cap J) - 1} \times \mathbb{A}^{\sharp J - \sharp(I \cap J)}](1 - \mathbb{L}) = -[C]\mathbb{L}^{\sharp J - \sharp(I \cap J)}(\mathbb{L} - 1)^{\sharp(I \cap J)} = 0 \pmod{(\mathbb{L} - 1)^2}.$$

If $\sharp(I \setminus J) = 0$, equivalently if $Z \subset D_i$ for every $i \in I$, then for every $i \in I$,

$$\tilde{C} \cap E_{0i}^\circ = \mathbb{P}_C^{\sharp J - 2} \setminus \bigcup_{j \in I \setminus \{i\}} H_j,$$

where H_j are coordinate hyperplanes of $\mathbb{P}_C^{\sharp J - 2}$. We have

$$A_i = [C][\mathbb{G}_m^{\sharp I - 2} \times \mathbb{A}^{\sharp J - \sharp I}](1 - \mathbb{L}) = -[C]\mathbb{L}^{\sharp J - \sharp I}(\mathbb{L} - 1)^{\sharp I - 1} = 0 \pmod{(\mathbb{L} - 1)^2}.$$

We thus have proved that $\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, \tilde{C}) = 0$ also when $d' \geq 3$, which completes the proofs of Proposition 2.1 and Theorem 1.4.

6. CLOSING COMMENTS

It is natural to try to refine $\tilde{S}(X)$ further by lifting it to $K_0(\text{Var}_k)_{\mathbb{Q}}/(\mathbb{L} - 1)^n$ for $n > 2$ and by adding extra terms of the form

$$c[D_H^\circ](1 - \mathbb{L})^{\sharp H - 1}$$

with $c \in \mathbb{Q}$, $H \subset I$, $\sharp H \geq 3$. However the author did not manage to find such a refinement.

The original invariant considered by Serre [11] and denoted by $i(X)$ was defined for a K -analytic manifold when the residue field k is finite, and lives in $\mathbb{Z}/(\sharp k - 1)$. There seems to be no counterpart of $\tilde{S}(X)$ in this context, at least in a naive way, because $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ is a field and the ideal generated by $(\sharp k - 1)^2$ in it is the entire field.

The author has no convincing explanation of the meaning of fractional coefficients appearing in the definitin of $\tilde{S}(X)$. However, as a possibly related work, we note that

also Denef and Loeser [4] previously considered motivic invariants with coefficients in \mathbb{Q} .

Nicaise and Sebag [10, Th. 5.4] gave a nice interpretation of the Euler characteristic representation of $S(X)$ in terms of cohomology of the generic fiber (see also [6] for another proof). It would be interesting to look for a similar interpretation of representations of $\tilde{S}(X)$ or $\tilde{S}(X)$ itself.

REFERENCES

- [1] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.*, 15(3):531–572 (electronic), 2002.
- [2] Dan Abramovich and Michael Temkin. Functorial factorization of birational maps for qc schemes in characteristic 0. arXiv:1606.08414.
- [3] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.
- [4] Jan Denef and François Loeser. Definable sets, motives and p -adic integrals. *J. Amer. Math. Soc.*, 14(2):429–469 (electronic), 2001.
- [5] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [6] Ehud Hrushovski and François Loeser. Monodromy and the Lefschetz fixed point formula. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(2):313–349, 2015.
- [7] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, (36):195–279, 1969.
- [8] Joseph Lipman. Desingularization of two-dimensional schemes. *Ann. Math. (2)*, 107(1):151–207, 1978.
- [9] François Loeser and Julien Sebag. Motivic integration on smooth rigid varieties and invariants of degenerations. *Duke Math. J.*, 119(2):315–344, 2003.
- [10] Johannes Nicaise and Julien Sebag. Motivic Serre invariants, ramification, and the analytic Milnor fiber. *Invent. Math.*, 168(1):133–173, 2007.
- [11] Jean-Pierre Serre. Classification des variétés analytiques p -adiques compactes. *Topology*, 3:409–412, 1965.
- [12] Michael Temkin. Desingularization of quasi-excellent schemes in characteristic zero. *Adv. Math.*, 219(2):488–522, 2008.
- [13] Michael Temkin. Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case. *Duke Math. J.*, 161(11):2207–2254, 2012.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN

E-mail address: takehikoyasuda@math.sci.osaka-u.ac.jp